

# MC SHANE'S IDENTITY IN RANK ONE SYMMETRIC SPACES

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**ABSTRACT.** In this paper we study McShane's identity in real and complex hyperbolic spaces and obtain various generalizations of the identity for representations of surface groups into the isometry groups of rank one symmetric spaces. Our methods unify most of the existing methods used in the existing literature for proving this class of identities.

## 1. INTRODUCTION

Greg McShane proved a striking identity for the lengths of simple closed geodesics for once-punctured hyperbolic tori in his PhD thesis [9] which he generalized later to cusped hyperbolic surfaces in [10]. A remarkable feature of this identity is that it involved the functions of the lengths of simple closed geodesics on the surface as opposed to the Selberg trace formula, which involved functions of the lengths of all primitive geodesics on the surface. Since then, there have been many generalizations of the original McShane's identity, and also alternative proofs; for example by Bowditch [4], [5], Akiyoshi-Miyachi-Sakuma [1], Mirzakhani [11] and Tan-Wong-Zhang [16], [17], [18].

The identities for hyperbolic surfaces with boundaries and/or cone points have found important applications by Mirzakhani in the computation of the Weil-Petersson volume of the moduli space of hyperbolic surfaces with boundaries and/or cone singularities ([11], [16]), developed further by her in [12] to give an alternative proof of the Kontsevich-Witten theorem on the Virasoro relations for the intersection numbers of tautological classes on the moduli space of stable curves

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as well as in [13] to study the asymptotic growth of the number of simple closed geodesics on hyperbolic surfaces. Indeed, the identities have, together with the Fenchel-Nielsen coordinates proved to be important tools in the study of the Weil-Petersson geometry of the moduli space of curves.

More recently, Labourie-McShane formulated the identity in terms of cross-ratios [8], thereby generalizing the identities to representations of surface groups into  $\mathrm{PSL}(n, \mathbb{R})$ . In this article, we use their idea to give the most general version of the identity for surfaces with boundary in rank one symmetric spaces in terms of the cross-ratio. More specifically, when the symmetric space is complex hyperbolic, we have  $\mathbb{C}$ -valued cross-ratios and the McShane's identity is a complex equation. For 3-dimensional real hyperbolic case, the cross-ratio is also  $\mathbb{C}$ -valued and we also recover some of known generalizations by Bowditch [4], and Akiyoshi-Miyachi-Sakuma [1].

Our main results in this paper are identities for hyperbolic mapping tori and complex hyperbolic manifolds using cross-ratio techniques, see Theorems 4.1 and 4.2. Recently, there has been much work on the representation spaces of surface groups into other Lie groups besides  $\mathrm{PSL}(2, \mathbb{R})$  and  $\mathrm{PSL}(2, \mathbb{C})$  and we expect that the generalized identities will eventually also play an important role in the understanding of the geometry of these spaces.

## 2. PRELIMINARIES

Let  $R$  be one of the four division rings. Due to the lack of commutativity for quaternions and octonions, we will not define the quaternion valued or octonion valued cross-ratios. There is a way to define exponential (hence logarithm) on quaternions, see for example [19] but it does not satisfy the usual rules like  $e^x e^y = e^{x+y}$ , so it is not useful to us. For those two division rings, we will define only real valued cross-ratios.

Let  $X$  be a rank one symmetric space in the unit ball model. For  $x, y \in X$ , the distance between these two points are:

$$\cosh(d(x, y)) = \frac{(|1 - \langle x, y \rangle|^2 + 2R\langle x, y \rangle)^{1/2}}{(1 - \langle x, x \rangle)^{1/2}(1 - \langle y, y \rangle)^{1/2}}$$

where  $R < v, w \rangle = \mathrm{Re}(v_1 \bar{v}_2)(w_2 \bar{w}_1) - \mathrm{Re}(\bar{v}_2 w_2)(\bar{w}_1 v_1)$  for the Cayley hyperbolic case and  $R < v, w \rangle = 0$  for other cases. See [14].

We define the real cross ratio in the unit ball model by:

$$[x, z, y, w]_{\mathbb{R}} = \frac{\langle \langle z, x \rangle \rangle \langle \langle w, y \rangle \rangle}{\langle \langle w, x \rangle \rangle \langle \langle z, y \rangle \rangle}$$

where  $\langle \langle x, y \rangle \rangle = (|1 - \langle x, y \rangle|^2 + 2R\langle x, y \rangle)^{1/2}$ .

We can rewrite the definition in terms of the generalized Heisenberg group  $N$  of  $X$ . Note that the ideal boundary of  $X$  is a one point compactification of  $N$  which is a two step Nilpotent group. The following is proved in [7] (note that the order of entries in the definition of the cross-ratio is changed for our convenience).

**Proposition 2.1.** *The real cross ratio above can be written as:*

$$[g_1, g_3, g_2, g_4]_{\mathbb{R}} = \frac{|g_3^{-1}g_1|^2|g_4^{-1}g_2|^2}{|g_4^{-1}g_1|^2|g_3^{-1}g_2|^2}$$

where  $x, y, z, w$  correspond to  $g_1, g_2, g_3, g_4$  in the Heisenberg group respectively.

The ideal boundary of rank one symmetric space has a so-called Gromov visual metric and it is equivalent to the left-invariant Heisenberg metric  $d(g, h) = |h^{-1}g|$ . We will use this metric when we talk about the Hölder structure. Note that in real hyperbolic case, the associated Heisenberg group is just  $\mathbb{R}^n$  and the real cross-ratio becomes

$$[X, Y, Z, W]_{\mathbb{R}} = \frac{|X - Y||Z - W|}{|X - W||Z - Y|}, \quad X, Y, Z, W \in \mathbb{R}^n.$$

Specifically, for real 2 or 3-dimensional case, one can define a real ( resp. complex) valued cross-ratio as  $[x, y, z, w] = \frac{(x-y)(z-w)}{(x-w)(z-y)}$  for  $x, y, z, w \in \mathbb{C}$ .

Now we define the  $\mathbb{C}$ -valued cross-ratio on the ideal boundary of complex hyperbolic manifold. If  $a, b, c, d$  are represented by  $z_1, z_2, z_3, z_4$  on the boundary of unit ball model, and  $\tilde{z}_1 = (z_1, 1), \tilde{z}_2 = (z_2, 1), \tilde{z}_3 = (z_3, 1), \tilde{z}_4 = (z_4, 1)$  are lifts in paraboloid model, then

$$[a, b, c, d] = \frac{\langle \tilde{z}_1, \tilde{z}_2 \rangle \langle \tilde{z}_3, \tilde{z}_4 \rangle}{\langle \tilde{z}_1, \tilde{z}_4 \rangle \langle \tilde{z}_3, \tilde{z}_2 \rangle}$$

where  $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle = \sum_{i=1}^{n-1} x_i \bar{y}_i - x_n \bar{y}_n$ . Since  $z_i$  are on the boundary of the unit ball model, their norm is 1 and so  $\langle \tilde{z}_i, \tilde{z}_i \rangle = 0$ .

The cross-ratio above satisfies:

- (1)  $[a, b, c, d] = 0$  iff  $a = b$  or  $c = d$ .
- (2)  $[a, b, c, d] = 1$  iff  $a = c$  or  $b = d$ .
- (3)  $[x, y, z, t] = [x, y, w, t][w, y, z, t]$
- (4)  $[x, y, z, t] = [x, y, z, w][x, w, z, t]$
- (5)  $[x, y, z, t] = [z, t, x, y]$
- (6)  $[x, y, z, t] = [z, y, x, t]^{-1}$
- (7)  $[x, y, z, t] = [x, t, z, y]^{-1}$
- (8)  $[x, y, z, t]_{\mathbb{R}} = |[x, y, z, t]|$

For a hyperbolic isometry  $\gamma$ , the period of  $\gamma$  is defined as

$$\ell(\gamma) = \log[\gamma^-, \gamma y, \gamma^+, y].$$

In rank one symmetric space, it is shown in [7] that

$$\operatorname{Re} \log[\gamma^-, \gamma y, \gamma^+, y] = \log |[\gamma^-, \gamma y, \gamma^+, y]| = l(\gamma)$$

where  $l(\gamma)$  is a translation length of  $\gamma$ . Henceforth, the cross-ratio is understood as  $\mathbb{C}$ -valued for complex hyperbolic space and 3-dimensional real hyperbolic space, and  $\mathbb{R}$ -valued otherwise.

On complex hyperbolic space  $\mathbf{H}_{\mathbb{C}}^n$ , a general reference is [6], but [15] is enough to follow this paper. If we choose a second Hermitian form on  $\mathbb{C}^{n,1}$ , for column vectors  $z = (z_1, \dots, z_{n+1})$  and  $w = (w_1, \dots, w_{n+1})$ ,

$$\langle z, w \rangle = w^* J z = z_1 \overline{w_{n+1}} + z_{n+1} \overline{w_1} + \sum_{j=2}^n z_j \overline{w_j}, \text{ where}$$

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and  $w^*$  is the Hermitian transpose of  $w$ .

We define the Siegel domain  $\mathfrak{S}$  of a complex hyperbolic space  $\mathbf{H}_{\mathbb{C}}^n$  by identifying points of  $\mathfrak{S}$  with their horospherical coordinates,  $z = (\zeta, v, u) \in \mathbb{C}^{n-1} \times \mathbb{R} \times \mathbb{R}_+$ . The boundary of  $\mathfrak{S}$  is given by  $H_0 \cup \{q_\infty\}$ , where  $q_\infty$  is a distinguished point at infinity and  $H_0 = \mathbb{C}^{n-1} \times \mathbb{R} \times \{0\}$ . Define a map  $\psi : \overline{\mathfrak{S}} \rightarrow \mathbb{P}\mathbb{C}^{n,1}$  by

$$\psi : (\zeta, v, u) \mapsto \begin{bmatrix} (-|\zeta|^2 - u + iv)/2 \\ \zeta \\ 1 \end{bmatrix} \text{ for } (\zeta, v, u) \in \overline{\mathfrak{S}} - \{q_\infty\}; \quad \psi : q_\infty \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then  $\psi$  maps  $\mathfrak{S}$  homeomorphically to the set of points  $z$  in  $\mathbb{P}\mathbb{C}^{n,1}$  with  $\langle z, z \rangle < 0$ , and maps  $\partial\mathfrak{S}$  homeomorphically to the set of points  $z$  in  $\mathbb{P}\mathbb{C}^{n,1}$  with  $\langle z, z \rangle = 0$ . We write  $\psi(\tilde{z}) = z$ .

The Bergman metric on  $\mathfrak{S}$  is given by the distance formula

$$\cosh^2 \left( \frac{\rho(\tilde{z}, \tilde{w})}{2} \right) = \frac{\langle z, w \rangle \langle w, z \rangle}{\langle z, z \rangle \langle w, w \rangle}.$$

The boundary of a complex hyperbolic space is identified with the one point compactification of the Heisenberg group. The  $(2n-1)$ -dimensional Heisenberg group  $\mathfrak{H}$  is  $\mathbb{C}^{n-1} \times \mathbb{R}$  with the group law

$$(\zeta_1, v_1) \diamond (\zeta_2, v_2) = (\zeta_1 + \zeta_2, v_1 + v_2 + 2\operatorname{Im}\langle \zeta_1, \zeta_2 \rangle)$$

where  $\langle \zeta_1, \zeta_2 \rangle$  is the standard positive-definite Hermitian form on  $\mathbb{C}^{n-1}$ .

The Heisenberg norm assigns to  $q = (\zeta, v)$  in  $\mathfrak{N}$  the nonnegative real number

$$|q| = (|\zeta|^4 + v^2)^{1/4}.$$

This gives rise to the Cygan metric on  $\mathfrak{N}$  by

$$\begin{aligned} \rho_0((\zeta_1, v_1), (\zeta_2, v_2)) &= |(\zeta_1, v_1)^{-1} \diamond (\zeta_2, v_2)| \\ &= ||\zeta_1 - \zeta_2|^2 - iv_1 + iv_2 - 2i\text{Im}\langle\zeta_1, \zeta_2\rangle|^{1/2}. \end{aligned}$$

The full group of holomorphic isometries of  $\mathbf{H}_{\mathbb{C}}^2$  is  $PU(2, 1)$ . Since  $PU(2, 1) = SU(2, 1)/\{I, \omega I, \omega^2 I\}$ , where  $\omega$  is a non-real cube root of unity, we may consider  $SU(2, 1)$  instead of  $PU(2, 1)$ . Then, as in the real hyperbolic isometry, we can classify isometries of complex hyperbolic space as loxodromic, parabolic, or elliptic by their fixed points. If  $A \in SU(2, 1)$  is a matrix representing a loxodromic isometry and  $e^\lambda$  is an attracting eigenvalue of  $A$ ,

$$A = \begin{bmatrix} e^\lambda & 0 & 0 \\ 0 & e^{\bar{\lambda}-\lambda} & 0 \\ 0 & 0 & e^{-\bar{\lambda}} \end{bmatrix},$$

where  $\lambda \in S = \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) > 0, \text{Im}(\lambda) \in (-\pi, \pi]\}$ .

In [15], Parker and Platis introduced the cross-ratio variety. Given distinct points  $z_1, z_2, z_3, z_4 \in \partial\mathbf{H}_{\mathbb{C}}^2$ , they defined

$$\mathbb{X}_1 = [z_4, z_2, z_3, z_1], \quad \mathbb{X}_2 = [z_4, z_3, z_2, z_1], \quad \mathbb{X}_3 = [z_4, z_3, z_1, z_2].$$

Then, the following identities hold.

**Proposition 2.2.** *(Proposition 5.2 in [15]) Let  $\mathbb{X}_1, \mathbb{X}_2$ , and  $\mathbb{X}_3$  be defined as above. Then,*

$$|\mathbb{X}_2|^2 = |\mathbb{X}_1||\mathbb{X}_3|,$$

$$2|\mathbb{X}_1|^2 \text{Re}(\mathbb{X}_3) = |\mathbb{X}_1|^2 + |\mathbb{X}_2|^2 + 1 - 2\text{Re}(\mathbb{X}_1 + \mathbb{X}_2).$$

An important property on complex cross-ratio worth taking note of for our purposes is the following.

**Lemma 2.3.** *The complex valued cross-ratio is Hölder if we fix three points  $x, z, t$  and  $y$  varies away from  $x$  and  $z$ , where  $x$  and  $z$  are not close to  $y$ .*

*Proof.* Since  $SU(2, 1)$  acts on 2-transitively on  $\mathbf{H}_{\mathbb{C}}^2$ , we may assume that  $x = 0$  and  $z = \infty$ . Let  $y_1 = (\zeta_1, v_1), y_2 = (\zeta_2, v_2)$  and  $t = (\zeta, v)$ .

Then as elements of  $\mathbb{PC}^{2,1}$ ,

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, z = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, y_1 = \begin{bmatrix} -|\zeta_1|^2 + iv_1 \\ \sqrt{2}\zeta_1 \\ 1 \end{bmatrix}, y_2 = \begin{bmatrix} -|\zeta_2|^2 + iv_2 \\ \sqrt{2}\zeta_2 \\ 1 \end{bmatrix}, t = \begin{bmatrix} -|\zeta|^2 + iv \\ \sqrt{2}\zeta \\ 1 \end{bmatrix}.$$

Then  $\langle z, t \rangle = \langle z, y_1 \rangle = \langle z, y_2 \rangle = 1$  and, for the convenience of calculation, let  $\zeta_1 = r_1 e^{i\theta_1}$ ,  $\zeta_2 = r_2 e^{i\theta_2}$ ,  $d := v_2 - v_1$ ,  $\theta := \theta_2 - \theta_1$  and

$K := |\langle x, t \rangle| = |-|\zeta|^2 - iv|$ . Since  $y_1$  and  $y_2$  are not close to  $x$  and  $z$ , there exist positive constants  $a$  and  $b$  such that  $a < \sqrt{r_i^4 + v_i^2} < b$ , for  $i = 1, 2$ . Then, obviously,  $r_i^2, |v_i| < b$ .

Let  $\rho := \rho_0(y_1, y_2)$ . By easy calculation,

$$\begin{aligned} \rho^4 &= |r_2^2 + r_1^2 - 2r_1 r_2 e^{i\theta} - id|^2 \\ &= (r_2^2 + r_1^2 - 2r_1 r_2 \cos \theta)^2 + (d + 2r_1 r_2 \sin \theta)^2. \end{aligned}$$

Hence,

$$\sqrt{(r_1^2 + r_2^2)^2 + d^2} - 2r_1 r_2 \leq \rho^2 \leq \sqrt{(r_1^2 + r_2^2)^2 + d^2} + 2r_1 r_2.$$

Furthermore,

$$\begin{aligned} A &:= |[x, y_1, z, t] - [x, y_2, z, t]| \\ &= \left| \frac{\langle x, y_1 \rangle \langle z, t \rangle}{\langle x, t \rangle \langle z, y_1 \rangle} - \frac{\langle x, y_2 \rangle \langle z, t \rangle}{\langle x, t \rangle \langle z, y_2 \rangle} \right| \\ &= \frac{1}{K} |\langle x, y_1 \rangle - \langle x, y_2 \rangle| \\ &= \frac{1}{K} |-r_1^2 - iv_1 + r_2^2 + iv_2| \\ &= \frac{1}{K} \sqrt{(r_2^2 - r_1^2)^2 + d^2}. \end{aligned}$$

Then,  $\rho^2 \geq \sqrt{KA^2 + 4r_1^2 r_2^2} - 2r_1 r_2$ , so  $\rho^4 + 4r_1^2 r_2^2 \rho^2 \geq KA^2$ . Therefore,

$$\begin{aligned} A &\leq \frac{\rho}{\sqrt{K}} (\rho^2 + 4r_1^2 r_2^2)^{1/2} \\ &\leq \frac{\rho}{\sqrt{K}} (\sqrt{(r_1^2 + r_2^2)^2 + d^2} + 2r_1 r_2 + 4r_1^2 r_2^2)^{1/2} \\ &\leq \frac{\rho}{\sqrt{K}} (\sqrt{r_1^4 + v_1^2 + r_2^4 + v_2^2 + 2r_1^2 r_2^2 + 2|v_1 v_2|} + 2r_1 r_2 + 4r_1^2 r_2^2)^{1/2} \\ &\leq \frac{\rho}{\sqrt{K}} (\sqrt{8b^2} + 2b + 4b^2)^{1/2} \\ &= C\rho, \end{aligned}$$

where  $C = (\frac{\sqrt{8}b + 2b + 4b^2}{K})$ .  $\square$

**Remark 2.4.** It is known that the real valued cross-ratio is Hölder. See [8].

### 3. FINITE TYPE HYPERBOLIC SURFACES

Let  $X$  be a Riemann surface with geodesic boundaries and punctures whose Euler number is negative. In this case, there is  $\Gamma \subset PSL(2, \mathbb{R})$  so that  $H_{\mathbb{R}}^2/\Gamma$  is an infinite volume hyperbolic surface so that if we truncate all the flaring ends along the geodesics we obtain  $X$ . If we give another such metric, these two hyperbolic metrics are quasi-isometric and there is a  $\pi_1(X)$ -equivariant Hölder homeomorphism between two limit sets. Usually  $\partial_{\infty}\pi_1(X)$  is identified with one of this limit set with inherited Hölder structure. Note that the universal cover  $\tilde{X}$  is exactly equal to the convex hull of the limit set and the ideal boundary of  $\tilde{X}$  is exactly  $\partial_{\infty}\pi_1(X)$ . Any element in  $\pi_1(X)$  acts on  $\partial_{\infty}\pi_1(X)$  as Hölder homeomorphisms.

Let  $\Gamma \subset PSL(2, \mathbb{R}) \subset G$  be a natural inclusion stabilizing totally geodesic Lagrangian  $H_{\mathbb{R}}^2$  in the rank one symmetric space  $X$  associated to  $G$ . Then it is known that there are type preserving small deformations  $\rho_t$  around  $\Gamma$ , i.e., deformations not stabilizing  $H_{\mathbb{R}}^2$  and whose convex core has a finite volume. For example, a quasifuchsian group in 3-dimensional hyperbolic case. In this case the limit set is still a Jordan curve in  $\partial X$  so that the induced map  $f_t$  from the limit set of  $\Gamma$  to the one of  $\rho_t$  is a Hölder homeomorphism [3]. If we identify  $\partial_{\infty}\pi_1(X)$  with the limit set of  $\Gamma$ , then there is a cross-ratio defined as

$$B_{\rho_t}(x, y, z, w) = [f_t x, f_t y, f_t z, f_t w]$$

where the right side cross-ratio is defined on  $\partial X$ . Note that if we fix 3 points, say  $x, z, w$ , then the function  $B(y) = B_{\rho_t}(x, y, z, w) : \partial_{\infty}\pi_1(X) \rightarrow \mathbb{R}, \mathbb{C}$  is Hölder when  $y$  varies on some domain away from  $x$  and  $z$  since it is a composition of two Hölder maps  $f_t$  and  $[f_t x, \cdot, f_t z, f_t w]$ .

Let  $\alpha$  denote a geodesic boundary of  $X$ . A pair of pants embedded in  $X$  with first boundary  $\alpha$  is an isotopy class of embeddings of a pair of pants  $P$  whose boundary is mapped to  $(\alpha, \beta, \gamma)$  so that  $\alpha\gamma\beta = 1$  in  $\pi_1(X)$ . Note here that the isotopy is allowed to rotate along  $\alpha$ , so  $(\alpha, \beta, \gamma)$  and  $(\alpha, \alpha^n\beta\alpha^{-n}, \alpha^n\gamma\alpha^{-n})$  represent the same isotopy class.

For any two points in  $\partial_{\infty}\pi_1(X)$ , one can connect them by a geodesic on  $\tilde{X}$  and project it down to  $X$ . If this geodesic is simple, we say the pair belongs to the Birman-Series set. Birman and Series showed that this set has zero Hausdorff dimension. Suppose  $\alpha$  is either boundary

or cusp of  $X$  and denote  $\alpha^+$  attracting point,  $\alpha^-$  repelling point of  $\alpha$  respectively. We give a canonical counterclockwise orientation on  $\partial_\infty \pi_1(X)$ .  $K_\alpha$  is a subset of  $\partial_\infty \pi_1(X) \setminus \{\alpha^+, \alpha^-\}$  consisting of  $t$  so that  $\{\alpha^+, t\}$  is a simple geodesic spiraling around  $\alpha$ . If  $\alpha$  is a cusp, a geodesic is spiraling around  $\alpha$  simply means that its end is shooting up to the cusp  $\alpha$ .  $K_\alpha^*$  is the subset of  $K_\alpha$ , excluding the  $\pi_1(X)$  orbit of  $\alpha^+$ .

**Definition 3.1.** For  $x, y \in K_\alpha^*$ , a gap  $(x, y) = \{t \in \partial_\infty \pi_1(X) | x < t < y \text{ in counterclockwise ordering}\}$  is such that  $(x, y) \cap K_\alpha^* = \emptyset$ .

So  $\partial_\infty \pi_1(X) \setminus K_\alpha^*$  is a disjoint union of gaps. It is known by [10, 11]

**Theorem 3.2.** *For a pair of pants  $(\alpha, \beta, \gamma)$ ,  $(\beta^+, \gamma^-)$  is a gap and if  $\beta$  is a boundary element, then  $(\beta^-, \beta^+)$  is a gap. Conversely every gap arises this way.*

Associated to these two types of gaps, one can associate gap functions

- (1) If  $P = (\alpha, \beta, \gamma)$  represents a pair of pants with first boundary  $\alpha$ , then the gap function is

$$G(P) = \log[\alpha^+, \gamma^-, \alpha^-, \beta^+]$$

- (2) If  $P = (\alpha, \beta, \gamma)$  represents a pair of pants so that  $\beta$  is also a boundary, then

$$G^r(P) = \log[\alpha^+, \beta^+, \alpha^-, \beta^-].$$

Note that by

$$\frac{[\alpha^+, \beta^+, \alpha^-, \zeta]}{[\alpha^+, \beta^-, \alpha^-, \zeta]} = [\alpha^+, \beta^+, \alpha^-, \beta^-]$$

for any  $\zeta \in \partial_\infty \pi_1(X)$ , we have

$$\log[\alpha^+, \beta^+, \alpha^-, \zeta] - \log[\alpha^+, \beta^-, \alpha^-, \zeta] = G^r(P).$$

A similar equation holds for the first type gap.

#### 4. APPLICATIONS TO REAL AND COMPLEX HYPERBOLIC SPACES

In this section we prove

**Theorem 4.1.** *Let  $\Gamma \subset SL(2, \mathbb{R}) = SO^0(2, 1)$  be a discrete group whose quotient of  $H_{\mathbb{R}}^2$  is a hyperbolic surface and its truncation of flaring ends along closed geodesics is a Riemann surface  $X$  with  $\alpha$  a boundary component. Let  $G$  be a rank one semi-simple Lie group associated*



with a division ring  $R = \mathbb{R}, \mathbb{C}$ . If  $\rho_t : \Gamma \rightarrow G$  is a type preserving deformation of  $\Gamma$  for small  $t$ , then the following generalized McShane's identity holds.

$$\ell(\alpha) = \sum_{P \in \mathcal{P}_\alpha} G(P) + \sum_{P \in \mathcal{S}_\alpha} G^r(P)$$

where  $\mathcal{P}_\alpha$  is the set of pants  $(\alpha, \beta, \gamma)$  so that  $(\alpha, \beta, \gamma) = (\alpha, \alpha^n \beta \alpha^{-n}, \alpha^n \gamma \alpha^{-n})$  and  $\mathcal{S}_\alpha$  is the set of pants  $(\alpha, \beta, \gamma)$  so that either  $\beta$  or  $\gamma$  is a boundary of  $X$ , say  $\beta$ . The gap functions are defined as

$$G(P) = \log[\alpha^+, \gamma^-, \alpha^-, \beta^+]$$

$$G^r(P) = \log[\alpha^+, \beta^+, \alpha^-, \beta^-].$$

*Proof.* For  $\rho_t$ , identify  $\partial_\infty \pi_1(X)$  with a limit set of  $\rho_t(\Gamma)$  via a homeomorphism. Note that when  $t = 0$ , the limit set lies on the round circle  $\partial H_{\mathbb{R}}^2$ . Define  $B : \partial_\infty \pi_1(X) \setminus \{\alpha^\pm\} \rightarrow R$ , where  $R = \mathbb{R}$  or  $\mathbb{C}$ , by

$$B(y) = [\alpha^+, y, \alpha^-, \zeta]$$

where  $\zeta \in \partial_\infty \pi_1(X) \setminus \{\alpha^\pm\}$  is a fixed reference point. Note that when  $y = \alpha^+$ ,  $B(y) = 0$  and  $y = \zeta$ ,  $B(y) = 1$  and  $y = \alpha^-$ ,  $B(y) = \infty$ . So the image of  $B$  lies on a curve in  $R$  emanating from the origin passing through 1 and diverging to  $\infty$ , which lies close to  $B(\partial_\infty \rho_0(\Gamma))$  when  $t$  is small. So take a line from the origin which does not intersect  $B(\partial_\infty \rho_0(\Gamma))$ , to define  $\log$ . By

$$[\alpha^+, \alpha(z), \alpha^-, \zeta] = [\alpha^+, z, \alpha^-, \zeta][\alpha^+, \alpha(z), \alpha^-, z],$$

$$\log B(\alpha(z)) = \log B(z) - \ell(\alpha).$$

This means that the action of  $\alpha$  on  $\partial_\infty \pi_1(X) \setminus \{\alpha^\pm\}$  and the image of  $B$  is related as above. So the fundamental domain  $D$  of  $\partial_\infty \pi_1(X) \setminus \{\alpha^\pm\}$  under the action of  $\alpha$  can be identified, via  $\log B$ , to a curve connecting  $\log B(z)$  and  $\log B(z) + \ell(\alpha)$  for any  $z$ . This map  $\log B$  is Hölder on  $D$ .

Also note that from

$$\frac{[\alpha^+, \gamma^-, \alpha^-, \zeta]}{[\alpha^+, \beta^+, \alpha^-, \zeta]} = [\alpha^+, \gamma^-, \alpha^-, \beta^+],$$

we get

$$\log B(\gamma^-) - \log B(\beta^+) = G(P).$$

Similar for  $G^r$ . This means that if  $(\beta^+, \gamma^-)$  is a gap, then their images under  $\log B$  differ by  $G(P)$  for a pants  $(\alpha, \beta, \gamma)$ .

In conclusion since  $\log B$  is Hölder,  $\log B(K_\alpha^*)$  has zero length and so if we sum up all  $\log B(\gamma^-) - \log B(\beta^+)$  and  $\log B(\beta^+) - \log B(\beta^-)$  for the gaps in the fundamental domain  $D$ , we should obtain  $\ell(\alpha)$ , i.e.,

$$\ell(\alpha) = \sum_{P \in \mathcal{P}_\alpha} G(P) + \sum_{P \in \mathcal{S}_\alpha} G^r(P).$$

□

When the representations are fuchsian, we recover original McShane-Mirzakhani identity. If  $\alpha$  represents a cusp, then  $G(P) = 0$  since  $\alpha^+ = \alpha^-$ . So we need an alternative approach. This case is dealt with in section 4.2 of [8]. Define

$$W_\alpha(s, t) = \frac{\partial_y \log[\alpha^+, s, y, t]}{\partial_y \log[\alpha^+, s_0, y, \alpha(s_0)]} \Big|_{y=\alpha^+}.$$

Then  $W_\alpha(s, t)$  is independent of  $s_0$  and satisfies

- (a)  $W_\alpha(\alpha(s), \alpha(t)) = W_\alpha(s, t)$
- (b)  $W_\alpha(s, \alpha(s)) = 1$
- (c)  $W_\alpha(s, u) = W_\alpha(s, t) + W_\alpha(t, u)$ .

Then one defines the cusp gap function for the pants  $P = (\alpha, \beta, \gamma)$  with the first boundary  $\alpha$  a cusp,

$$W(P) = W_\alpha(\gamma^-, \beta^+),$$

and if  $\beta$  is either peripheral or a cusp then

$$W^r(P) = W_\alpha(\beta^+, \beta^-).$$

Then we get:

**Theorem 4.2.** *If  $\alpha$  represents a cusp, then*

$$1 = \sum_{P \in \mathcal{P}_\alpha} W(P) + \sum_{P \in \mathcal{S}_\alpha} W^r(P).$$

*Proof.* One define a similar map  $B : \partial_\infty \pi_1(X) \setminus \{\alpha^+\} \rightarrow \mathbb{R} \text{ (or } \mathbb{C})$  by

$$B(y) = W_\alpha(s, y),$$

where  $s \in \partial_\infty \pi_1(X) \setminus \{\alpha^+\}$  is a fixed point. By the equations (b) and (c),

$$B(\alpha(y)) - B(y) = W_\alpha(s, \alpha(y)) - W_\alpha(s, y) = W_\alpha(y, \alpha(y)) = 1.$$

In rank one case, it is easy to check that  $B$  is still Hölder for geometrically finite case, so the sum of gaps between  $y$  and  $\alpha(y)$  is equal to 1, i.e.,

$$1 = \sum_{P \in \mathcal{P}_\alpha} W(P) + \sum_{P \in \mathcal{S}_\alpha} W^r(P).$$

□

This cuspidal case has many corollaries which cover known results so far. We will discuss this in next section.

## 5. APPLICATIONS

5.1. *PSL(2, ℂ) case.* The cross-ratio on  $\partial H_{\mathbb{R}}^3 = \hat{\mathbb{C}}$  is in most simple form

$$[a, b, c, d] = \frac{(a-b)(c-d)}{(a-d)(c-b)}$$

for  $a, b, c, d \in \mathbb{C}$ . This cross-ratio has another properties which does not share with other cases. Following sections will make use of this extra properties strongly. Any hyperbolic isometry  $\gamma$  can be conjugated into a form, whose action on  $\hat{\mathbb{C}}$  is

$$z \rightarrow r e^{i\theta} z, \quad r > 1$$

so that  $\log r + i\theta$  is called a complex length of  $\gamma$ . This hyperbolic isometry has the repelling fixed point 0 and the attracting fixed point  $\infty$ . Then the period  $\ell(\gamma)$  of  $\gamma$  is given by

$$\log(r e^{i\theta})$$

which is a complex length of  $\gamma$ .

5.1.1. *Explicit calculation.* Let  $P = (\alpha, \beta, \gamma)$  be a pair of pants so that  $\alpha\gamma\beta = 1$ . Then the shear coordinates for  $P$  is

$$A = -[\gamma^-, \alpha^+, \beta^+, \gamma^{-1}(\alpha^+)] = -[\gamma^-, \alpha^+, \beta^+, \beta(\alpha^+)]$$

$$B = -[\alpha^+, \beta^+, \gamma^-, \alpha^{-1}(\beta^+)]$$

$$C = -[\beta^+, \gamma^-, \alpha^+, \beta^{-1}(\gamma^-)].$$

A direct calculation shows that  $W(P) = [\alpha(\gamma^-), \alpha^+, \beta^+, \gamma^-]$  using  $s_0 = \gamma^-$  in the definition of  $W_\alpha$ . By manipulating the cross-ratio we get

$$W(P) = \frac{1}{1 + e^{\frac{\ell(\beta) + \ell(\gamma)}{2}}}.$$

Similarly

$$W^r(P) = \frac{\sinh \frac{\ell(\beta)}{2}}{\cosh \frac{\ell(\gamma)}{2} + \cosh \frac{\ell(\beta)}{2}}.$$

5.1.2. *quasifuchsian case.* Let  $X$  be a finite volume complete hyperbolic surface with  $\alpha$  a cusp. Then original McShane's identity reads:

$$\sum_{\beta, \gamma} \frac{2}{1 + e^{\frac{l(\beta) + l(\gamma)}{2}}} = 1,$$

where  $\beta \cup \gamma$  together with  $\alpha$  bounds a pair of pants. In our notation,  $(\alpha, \beta, \gamma)$  and  $(\alpha, \gamma, \beta)$  are different pairs of pants so that the factor 2 appears in the summand. Our result in Theorem 4.2 shows that this equality holds for any type preserving quasifuchsian representations where the real length  $l(\beta)$  is replaced by its complex length  $\ell(\beta)$ . This is obtained by [1] (Theorem 2.2) by a different method.

5.1.3. *Mapping torus case.* Let  $M$  be a complete hyperbolic 3-manifold which fiber over a circle with a punctured torus fibre  $T$ . By Thurston this manifold is a mapping torus whose monodromy is a pseudo-anasov map  $\phi$  on  $T$ . Then a limit set of a subgroup  $\Gamma_T$  corresponding to  $T$  is whole Riemann sphere  $\hat{\mathbb{C}}$ . Let  $\alpha$  denote the cusp in  $\pi_1(T)$  so that  $\alpha^+ = \alpha^-$ . Take a sequence of quasifuchsian groups  $Q_n = (\phi^{-n}X, \phi^nX)$  which converges to  $\Gamma_T$ . For each  $n$ , we have  $\sum_{\beta} \frac{2}{1 + e^{\ell(\beta)}} = 1$ . Since  $\beta$  and  $\phi^i(\beta)$  represent the same loop in  $M$ ,  $\ell(\beta) = \ell(\phi^i(\beta))$  for all  $i$  in  $M$ . This implies that for all  $Q_n$

$$\sum_{\beta \in \mathcal{S}/\phi} \sum_{i \in \mathbb{Z}} \frac{2}{1 + e^{\ell(\phi^i \beta)}} = 1,$$

hence we must have in  $M$

$$\sum_{\mathcal{S}/\phi} \frac{2}{1 + e^{\ell(\beta)}} = 0,$$

where  $\mathcal{S}$  denote the set of simple closed curves in  $T$  so that  $\mathcal{S}/\phi$  is the set of simple closed geodesics in  $M$ . This is the original form of the result by [5]. A similar result obtained by [1] also holds by our unified approach.

Let  $M$  be a mapping torus with fiber  $F$  a general punctured surface glued by a pseudoanasov map  $\phi$ . If  $\mathcal{S}$  denotes the set of free homotopy class of simple closed curves on  $F$ , then  $\mathcal{S}/\phi$  denotes the set of free homotopy class of simple closed curves on  $M$  coming from the fiber  $F$ .

Then  $Q_n \rightarrow Q_\infty$  strongly where  $Q_\infty$  corresponds to the fiber group of  $M$ . Note that  $\phi$  acts on  $Q_\infty$  as an isometry to give  $M = Q_\infty/\phi$ . Let  $\ell_n(\gamma)$  be the complex length of  $\gamma \in \pi_1(F)$  in  $Q_n$ . Then we will show that

**Proposition 5.1.** *For hyperbolic mapping torus  $M$  with a fiber  $F$  once punctured surface with a puncture  $\alpha$ ,*

$$\sum_{P=(\alpha,\beta,\gamma), \beta,\gamma \in \mathcal{S}/\phi} \frac{1}{1 + e^{\frac{\ell_M(\beta) + \ell_M(\gamma)}{2}}} = 0.$$

*Proof.* By geometric convergence, for any compact exhaustion  $K_i \subset Q_\infty$  of  $Q_\infty$ , there is  $N_i$  such that there exists an smooth embedding

$$f_n^i : K_i \rightarrow Q_n, \quad n > N_i$$

whose quasi-geodesic constant  $1 \leq C_n^i < 2$ . Then for any  $\gamma \in \mathcal{S}$  whose geodesic representative is in  $K_i$ ,

$$(1) \quad \frac{1}{2}l_{Q_n}(\gamma) \leq l_M(\gamma) \leq 2l_{Q_n}(\gamma), \quad n > N_i.$$

Let  $\mathcal{S}_{K_i}$  be the set of  $\gamma \in \mathcal{S}$  whose geodesic representative lies in  $K_i$ . By geometric convergence, such  $\gamma$  has a geodesic representative in a small neighborhood of  $f_n^i(K_i)$  in  $Q_n$  for  $n > N_i$ .

Now fix  $K_i$  for some  $i$ . For a fixed  $n > N_i$ , since  $Q_n$  is quasifuchsian there are  $k > 1$  and a fuchsian group  $\rho_0$  so that

$$\frac{1}{k}l_{\rho_0}(\gamma) \leq l_{Q_n}(\gamma) \leq kl_{\rho_0}(\gamma), \quad \forall \gamma.$$

In conclusion by Equation (1) there exist  $k > 1$  and a fuchsian group  $\rho_0$  so that

$$\frac{1}{k}l_{\rho_0}(\gamma) \leq l_M(\gamma) \leq kl_{\rho_0}(\gamma), \quad \forall \gamma \in \mathcal{S}_{K_i}.$$

Then again by Equation (1), abusing notation with the same  $k$ ,

$$(2) \quad \frac{1}{k}l_{\rho_0}(\gamma) \leq l_{Q_n}(\gamma) \leq kl_{\rho_0}(\gamma), \quad \forall \gamma, \forall n > N_i \text{ and } n = \infty$$

In [2], it is shown that for  $F = H^2/\rho_0$ , there exists  $A$  consisting of finitely many mutually disjoint simple complete geodesics each of which joins punctures of  $F$  such that for  $\gamma \in \mathcal{S}$ , if  $||\gamma||$  is the intersection number of  $\gamma$  and  $A$ , then there are  $c$  and a polynomial  $P(n)$  such that  $l_{\rho_0}(\gamma) \geq c||\gamma||$  and  $|S(n) = \{\gamma \in \mathcal{S} | ||\gamma|| = n\}| \leq P(n)$ . Then for any  $n > N_i$  and  $n = \infty$ ,

$$l_{Q_n}(\gamma) \geq \frac{1}{k}l_{\rho_0}(\gamma) \geq \frac{c}{k}||\gamma|| = \frac{c}{k}n, \quad \forall \gamma \in S(n) \cap \mathcal{S}_{K_i}.$$

For a pant  $P$  whose boundaries are  $\alpha, \beta, \gamma$  with  $\alpha$  a fixed cusp,

$$|W(P)| = \left| \frac{1}{1 + e^{\frac{\ell(\beta) + \ell(\gamma)}{2}}} \right| \leq \frac{1}{|e^{\frac{\ell(\beta) + \ell(\gamma)}{2}}| - 1}$$

$$\leq \frac{1}{e^{\frac{l(\beta)+l(\gamma)}{2}} - 1} \leq \frac{1}{e^{\frac{1}{2}l(\beta)} - 1} \frac{1}{e^{\frac{1}{2}l(\gamma)}}.$$

By inequality (2), for any  $n > N_i$  and  $n = \infty$ , since there are only finitely many  $S_0$  of  $\gamma$  so that  $l_{\rho_0}(\gamma) < 2k \log 2$ , except finitely many  $\gamma$ 's,  $l_{Q_n}(\gamma) \geq 2 \log 2$ . Hence outside  $S_0$

$$\frac{1}{e^{\frac{1}{2}l(\beta)} - 1} \leq 2 \frac{1}{e^{\frac{1}{2}l(\beta)}}.$$

Note that for any  $n > N_i$  and  $n = \infty$ ,

$$\sum_{\gamma \in \mathcal{S}_{K_i}} \frac{1}{e^{\frac{1}{2}l(\gamma)}} = \sum_{i=1}^{\infty} \sum_{\gamma \in \mathcal{S}_{K_i} \cap S(n)} \frac{1}{e^{\frac{1}{2}l(\gamma)}} \leq \sum_{i=1}^{\infty} \frac{P(n)}{e^{\frac{cn}{2k}}} = P_i < \infty.$$

Let  $\mathcal{P}_{\alpha}^i$  be the set of pairs of pants  $P = (\alpha, \beta, \gamma)$  so that  $\beta, \gamma \in S_{K_i}$ .

Given  $\epsilon > 0$ , choose a finite set  $\mathcal{S}_{K_i}^{\epsilon}$  containing  $S_0$  so that for any  $n > N_i, n = \infty$ ,

$$\sum_{\mathcal{S}_{K_i} \setminus \mathcal{S}_{K_i}^{\epsilon}} \frac{1}{e^{\frac{1}{2}l_{Q_n}(\gamma)}} < \epsilon.$$

Let  $\mathcal{P}_{\alpha}^{\epsilon}$  be a subset of  $\mathcal{P}_{\alpha}^i$  consisting of  $P$ 's so that  $\beta, \gamma \in \mathcal{S}_{K_i}^{\epsilon}$ . Then for any  $n > N_i$  and  $n = \infty$ ,

$$\begin{aligned} \sum_{P \in \mathcal{P}_{\alpha}^i} |W(P)| &\leq \sum_{P \in \mathcal{P}_{\alpha}^{\epsilon}} |W(P)| + \sum_{P \in \mathcal{P}_{\alpha}^i \setminus \mathcal{P}_{\alpha}^{\epsilon}} |W(P)| \\ &\leq \sum_{P \in \mathcal{P}_{\alpha}^{\epsilon}} |W(P)| + \sum_{P \in \mathcal{P}_{\alpha}^i \setminus \mathcal{P}_{\alpha}^{\epsilon}} 2 \frac{1}{e^{\frac{1}{2}l_{Q_n}(\beta)}} \frac{1}{e^{\frac{1}{2}l_{Q_n}(\gamma)}} \\ &\leq \sum_{P \in \mathcal{P}_{\alpha}^{\epsilon}} |W(P)| + 2 \left( \sum_{\mathcal{S}_{K_i} \setminus \mathcal{S}_{K_i}^{\epsilon}} \frac{1}{e^{\frac{1}{2}l_{Q_n}(\beta)}} \right) \left( \sum_{\mathcal{S}_{K_i}} \frac{1}{e^{\frac{1}{2}l_{Q_n}(\gamma)}} \right) \\ &\leq \sum_{P \in \mathcal{P}_{\alpha}^{\epsilon}} |W(P)| + 2\epsilon P_i. \end{aligned}$$

Since  $\mathcal{P}_{\alpha}^{\epsilon}$  is a finite set, this sum is uniformly bounded for all  $n > N_i$  and  $n = \infty$ . Hence for a given  $\epsilon$ , we can choose a finite set  $\mathcal{P}_{\alpha, \epsilon}^i \subset \mathcal{P}_{\alpha}^i$  so that for any  $n > N_i, n = \infty$

$$\left| \sum_{\mathcal{P}_{\alpha}^i \setminus \mathcal{P}_{\alpha, \epsilon}^i} W_n(P) \right| < \epsilon,$$

where  $W_n$  is measured in  $Q_n$ . Since  $\mathcal{P}_{\alpha, \epsilon}^i$  is a finite set, we can choose  $n_i > N_i$  so that

$$\left| \sum_{\mathcal{P}_{\alpha, \epsilon}^i} W_{\infty}(P) - \sum_{\mathcal{P}_{\alpha, \epsilon}^i} W_{n_i}(P) \right| < \epsilon,$$

hence

$$(3) \quad \left| \sum_{\mathcal{P}_\alpha^i} W_\infty(P) - \sum_{\mathcal{P}_\alpha^i} W_{n_i}(P) \right| < 3\epsilon.$$

Since  $M = Q_\infty/\phi$ , let  $K = K_0$  be the fundamental domain of  $M$  in  $Q_\infty$  and  $K_i$  is the union of  $K \cup \phi(K) \cup \dots \cup \phi^{i-1}(K)$ . Note that  $\mathcal{S}/\phi$  is identified with the free homotopy classes of simple loops in  $M$ . Since  $\phi$  is an isometry of  $Q_\infty$ ,  $l_{Q_n}(\gamma) = l_{Q_n}(\phi^j \gamma)$  for any  $j$ , hence

$$\sum_{\mathcal{P}_\alpha^i} W_\infty(P) = i \sum_{\mathcal{P}_\alpha^0} W_\infty(P).$$

For a given  $\epsilon$ , by Equation (3)

$$\left| \sum_{\mathcal{P}_\alpha^i} W_{n_i}(P) - i \sum_{\mathcal{P}_\alpha^0} W_\infty(P) \right| < 3\epsilon.$$

As  $i \rightarrow \infty$ ,  $P_\alpha^i \rightarrow P_\alpha$  and the left term  $|\sum_{\mathcal{P}_\alpha^i} W_{n_i}(P)|$  is bounded above near 1, hence

$$\sum_{P=(\alpha,\beta,\gamma), \beta,\gamma \in \mathcal{S}/\phi} \frac{2}{1 + e^{\frac{\ell_{\mathcal{M}(\beta)} + \ell_{\mathcal{M}(\gamma)}}{2}}} = 0.$$

□

**5.2.  $PSU(2,1)$  case.** This is the case where no previous results exist. Our chosen complex valued cross-ratio is: for  $z_1, z_2, z_3, z_4$  on the boundary of unit ball model representing four points  $a, b, c, d$ , and  $\tilde{z}_1 = (z_1, 1), \tilde{z}_2 = (z_2, 1), \tilde{z}_3 = (z_3, 1), \tilde{z}_4 = (z_4, 1)$  lifts in paraboloid model, then

$$[a, b, c, d] = \frac{\langle \tilde{z}_1, \tilde{z}_2 \rangle \langle \tilde{z}_3, \tilde{z}_4 \rangle}{\langle \tilde{z}_1, \tilde{z}_4 \rangle \langle \tilde{z}_3, \tilde{z}_2 \rangle}.$$

A general hyperbolic isometry in  $H_{\mathbb{C}}^2$  is in the form fixing 0 and  $\infty$ :

$$[t, z] \rightarrow [r^2 t, r e^{i\theta} z]$$

in Heisenberg coordinates. A direct calculation shows that the period of such a hyperbolic isometry  $\gamma$  is

$$\ell(\gamma) = \log r^2$$

which is a real translation length of  $\gamma$ . Somehow the period does not capture the rotational part. We want to derive formulae for cuspidal and boundary loop case as in fuchsian case.

5.2.1. *Gap functions.* Let  $P = (\alpha, \beta, \gamma)$  be a pair of pants so that  $\alpha\gamma\beta = 1$ . We may normalize so that  $\alpha$  fixes  $\infty$  and 0, i.e.  $\alpha^+ = \infty$  and  $\alpha^- = 0$ . Then, as a matrix point of view,

$$\alpha = E(\lambda) = \begin{bmatrix} e^\lambda & 0 & 0 \\ 0 & e^{\bar{\lambda}-\lambda} & 0 \\ 0 & 0 & e^{-\bar{\lambda}} \end{bmatrix},$$

where  $\lambda \in S$  and  $e^\lambda$  is an attracting eigenvalue of  $\alpha$ . Also for  $Q, R \in SU(2, 1)$  and  $\mu, \nu \in S$ , we can write

$$\gamma = QE(\mu)Q^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \begin{bmatrix} e^\mu & 0 & 0 \\ 0 & e^{\bar{\mu}-\mu} & 0 \\ 0 & 0 & e^{-\bar{\mu}} \end{bmatrix} \begin{bmatrix} \bar{j} & \bar{f} & \bar{c} \\ \bar{h} & \bar{e} & \bar{b} \\ \bar{g} & \bar{d} & \bar{a} \end{bmatrix},$$

$$\beta = \gamma^{-1}\alpha^{-1} = RE(\nu)R^{-1} = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & j' \end{bmatrix} \begin{bmatrix} e^\nu & 0 & 0 \\ 0 & e^{\bar{\nu}-\nu} & 0 \\ 0 & 0 & e^{-\bar{\nu}} \end{bmatrix} \begin{bmatrix} \bar{j}' & \bar{f}' & \bar{c}' \\ \bar{h}' & \bar{e}' & \bar{b}' \\ \bar{g}' & \bar{d}' & \bar{a}' \end{bmatrix},$$

where  $e^\mu$  and  $e^\nu$  are attracting eigenvalues of  $\gamma$  and  $\beta$  respectively. Furthermore, using some identities from  $QQ^{-1} = I$ , a direct calculation shows the following Lemma.

**Lemma 5.2.** (*Lemma 6.3 in [15]*)

$$\beta = \begin{bmatrix} e^{\bar{\nu}-\nu} + a'\bar{j}'\sigma(\nu) + c'\bar{g}'\sigma(-\bar{\nu}) & a'\bar{f}'\sigma(\nu) + c'\bar{d}'\sigma(-\bar{\nu}) & a'\bar{c}'\sigma(\nu) + c'\bar{a}'\sigma(-\bar{\nu}) \\ d'\bar{j}'\sigma(\nu) + f'\bar{g}'\sigma(-\bar{\nu}) & e^{\bar{\nu}-\nu} + d'\bar{f}'\sigma(\nu) + f'\bar{d}'\sigma(-\bar{\nu}) & d'\bar{c}'\sigma(\nu) + f'\bar{a}'\sigma(-\bar{\nu}) \\ g'\bar{j}'\sigma(\nu) + j'\bar{g}'\sigma(-\bar{\nu}) & g'\bar{f}'\sigma(\nu) + j'\bar{d}'\sigma(-\bar{\nu}) & e^{\bar{\nu}-\nu} + g'\bar{c}'\sigma(\nu) + j'\bar{a}'\sigma(-\bar{\nu}) \end{bmatrix},$$

where  $\sigma(\nu) = e^\nu - e^{\bar{\nu}-\nu}$ .

Also we can write  $\beta$  as the following, using the identity  $\alpha\gamma\beta = 1$ .

**Lemma 5.3.** *Let  $\alpha$ ,  $\gamma$  and  $\beta$  be defined as above. Then*

$$\beta = \begin{bmatrix} e^{-\lambda}(e^{\mu-\bar{\mu}} + a\bar{j}\sigma(-\mu) + c\bar{g}\sigma(\bar{\mu})) & e^{\lambda-\bar{\lambda}}(a\bar{f}\sigma(-\mu) + c\bar{d}\sigma(\bar{\mu})) & e^{\bar{\lambda}}(a\bar{c}\sigma(-\mu) + c\bar{a}\sigma(\bar{\mu})) \\ e^{-\lambda}(d\bar{j}\sigma(-\mu) + f\bar{g}\sigma(\bar{\mu})) & e^{\lambda-\bar{\lambda}}(e^{\mu-\bar{\mu}} + d\bar{f}\sigma(-\mu) + f\bar{d}\sigma(\bar{\mu})) & e^{\bar{\lambda}}(d\bar{c}\sigma(-\mu) + f\bar{a}\sigma(\bar{\mu})) \\ e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu})) & e^{\lambda-\bar{\lambda}}(g\bar{f}\sigma(-\mu) + j\bar{d}\sigma(\bar{\mu})) & e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})) \end{bmatrix}.$$

*Proof.* The proof is direct from the following calculations.

$$\begin{aligned} \beta &= \gamma^{-1}\alpha^{-1} = QE(-\mu)Q^{-1}E(-\lambda) \\ &= \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \begin{bmatrix} e^{-\mu} & 0 & 0 \\ 0 & e^{\mu-\bar{\mu}} & 0 \\ 0 & 0 & e^{\bar{\mu}} \end{bmatrix} \begin{bmatrix} \bar{j} & \bar{f} & \bar{c} \\ \bar{h} & \bar{e} & \bar{b} \\ \bar{g} & \bar{d} & \bar{a} \end{bmatrix} \begin{bmatrix} e^{-\lambda} & 0 & 0 \\ 0 & e^{\lambda-\bar{\lambda}} & 0 \\ 0 & 0 & e^{\bar{\lambda}} \end{bmatrix} \end{aligned}$$



$$\begin{aligned}
&= \begin{bmatrix} e^{-\lambda}(e^{-\mu}a\bar{j} + e^{\mu-\bar{\mu}}b\bar{h} + e^{\bar{\mu}}c\bar{g}) & e^{\lambda-\bar{\lambda}}(e^{-\mu}a\bar{f} + e^{\mu-\bar{\mu}}b\bar{e} + e^{\bar{\mu}}c\bar{d}) & e^{\bar{\lambda}}(e^{-\mu}a\bar{c} + e^{\mu-\bar{\mu}}b\bar{b} + e^{\bar{\mu}}c\bar{a}) \\ e^{-\lambda}(e^{-\mu}d\bar{j} + e^{\mu-\bar{\mu}}e\bar{h} + e^{\bar{\mu}}f\bar{g}) & e^{\lambda-\bar{\lambda}}(e^{-\mu}d\bar{f} + e^{\mu-\bar{\mu}}e\bar{e} + e^{\bar{\mu}}f\bar{d}) & e^{\bar{\lambda}}(e^{-\mu}d\bar{c} + e^{\mu-\bar{\mu}}e\bar{b} + e^{\bar{\mu}}f\bar{a}) \\ e^{-\lambda}(e^{-\mu}g\bar{j} + e^{\mu-\bar{\mu}}h\bar{h} + e^{\bar{\mu}}j\bar{g}) & e^{\lambda-\bar{\lambda}}(e^{-\mu}g\bar{f} + e^{\mu-\bar{\mu}}h\bar{e} + e^{\bar{\mu}}j\bar{d}) & e^{\bar{\lambda}}(e^{-\mu}g\bar{c} + e^{\mu-\bar{\mu}}h\bar{b} + e^{\bar{\mu}}j\bar{a}) \end{bmatrix} \\
&= \begin{bmatrix} e^{-\lambda}(e^{\mu-\bar{\mu}} + a\bar{j}\sigma(-\mu) + c\bar{g}\sigma(\bar{\mu})) & e^{\lambda-\bar{\lambda}}(a\bar{f}\sigma(-\mu) + c\bar{d}\sigma(\bar{\mu})) & e^{\bar{\lambda}}(a\bar{c}\sigma(-\mu) + c\bar{a}\sigma(\bar{\mu})) \\ e^{-\lambda}(d\bar{j}\sigma(-\mu) + f\bar{g}\sigma(\bar{\mu})) & e^{\lambda-\bar{\lambda}}(e^{\mu-\bar{\mu}} + d\bar{f}\sigma(-\mu) + f\bar{d}\sigma(\bar{\mu})) & e^{\bar{\lambda}}(d\bar{c}\sigma(-\mu) + f\bar{a}\sigma(\bar{\mu})) \\ e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu})) & e^{\lambda-\bar{\lambda}}(g\bar{f}\sigma(-\mu) + j\bar{d}\sigma(\bar{\mu})) & e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})) \end{bmatrix}.
\end{aligned}$$

For the last equality, we use six identities which are from  $QQ^{-1} = I$ , for example  $1 = a\bar{j} + b\bar{h} + c\bar{g}$  for the top left-hand entry.  $\square$

By comparing above two lemmas, we get the following proposition, which will be used later.

**Proposition 5.4.**

$$\frac{\bar{a}'}{\bar{g}'} = \frac{e^{-\bar{\lambda}}(e^{\bar{\mu}-\mu} + \bar{a}j\sigma(-\bar{\mu}) + \bar{c}g\sigma(\mu)) + e^{-\bar{\nu}}e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})) - e^{\nu-\bar{\nu}} - e^{-\nu}}{e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu}))}.$$

*Proof.* The left-bottom entry of  $\beta$  is

$$g'\bar{j}'\sigma(\nu) + j'\bar{g}'\sigma(-\bar{\nu}) = e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu}))$$

By conjugating, we get

$$g'\bar{j}'\sigma(-\nu) + j'\bar{g}'\sigma(\bar{\nu}) = e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu}))$$

We can rewrite these two equations as follows, using  $\sigma(-\nu) = -e^{-\bar{\nu}}\sigma(\nu)$ .

$$g'\bar{j}'e^{-\bar{\nu}}\sigma(\nu) + j'\bar{g}'e^{-\bar{\nu}}\sigma(-\bar{\nu}) = e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu})),$$

$$-g'\bar{j}'e^{-\bar{\nu}}\sigma(\nu) + j'\bar{g}'\sigma(\bar{\nu}) = e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu})).$$

By adding them, we have

$$(\sigma(\bar{\nu}) + e^{-\bar{\nu}}\sigma(-\bar{\nu}))j'\bar{g}' = e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu})).$$

Hence,

$$j' = \frac{1}{\bar{g}'} \frac{e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu}))}{(1 - e^{-(\nu+\bar{\nu})})\sigma(\bar{\nu})}.$$

Similarly, if we compare the right-top entry of  $\beta$ , we get

$$c' = \frac{1}{\bar{a}'} \frac{e^{\lambda}(a\bar{c}\sigma(\mu) + c\bar{a}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{\bar{\lambda}}(a\bar{c}\sigma(-\mu) + c\bar{a}\sigma(\bar{\mu}))}{(1 - e^{-(\nu+\bar{\nu})})\sigma(\bar{\nu})}.$$

If we substitute these  $j'$  and  $c'$  to the left-top entry and right-bottom entry in Lemma 5.2, then we have

$$\begin{aligned} & e^{\bar{\nu}-\nu} + \frac{a' e^{-\lambda}(\bar{g}j\sigma(\bar{\mu}) + \bar{j}g\sigma(-\mu)) + e^{-\nu}e^{-\bar{\lambda}}(\bar{g}j\sigma(-\bar{\mu}) + \bar{j}g\sigma(\mu))}{1 - e^{-(\nu+\bar{\nu})}} \\ & - e^{-\nu} \frac{\bar{g}' e^{\lambda}(a\bar{c}\sigma(\mu) + \bar{c}a\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{\bar{\lambda}}(a\bar{c}\sigma(-\mu) + \bar{c}a\sigma(\bar{\mu}))}{1 - e^{-(\nu+\bar{\nu})}} \\ & = e^{-\lambda}(e^{\mu-\bar{\mu}} + a\bar{j}\sigma(-\mu) + c\bar{g}\sigma(\bar{\mu})) \end{aligned}$$

and

$$\begin{aligned} & e^{\bar{\nu}-\nu} + \frac{g' e^{\bar{\lambda}}(\bar{a}c\sigma(\bar{\mu}) + \bar{c}a\sigma(-\mu)) + e^{-\nu}e^{\lambda}(\bar{a}c\sigma(-\bar{\mu}) + \bar{c}a\sigma(\mu))}{1 - e^{-(\nu+\bar{\nu})}} \\ & - e^{-\nu} \frac{\bar{a}' e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + \bar{j}g\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + \bar{j}g\sigma(\bar{\mu}))}{1 - e^{-(\nu+\bar{\nu})}} \\ & = e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})). \end{aligned}$$

Now we conjugate the first one and multiply  $e^{-\bar{\nu}}$  to the second one, then they are as follows respectively.

$$\begin{aligned} & e^{\nu-\bar{\nu}} - e^{-\bar{\nu}} g' \frac{e^{\bar{\lambda}}(\bar{a}c\sigma(\bar{\mu}) + \bar{c}a\sigma(-\mu)) + e^{-\nu}e^{\lambda}(\bar{a}c\sigma(-\bar{\mu}) + \bar{c}a\sigma(\mu))}{1 - e^{-(\nu+\bar{\nu})}} \\ & + \frac{\bar{a}' e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + \bar{j}g\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + \bar{j}g\sigma(\bar{\mu}))}{1 - e^{-(\nu+\bar{\nu})}} \\ & = e^{-\bar{\lambda}}(e^{\bar{\mu}-\mu} + \bar{a}j\sigma(-\bar{\mu}) + \bar{c}g\sigma(\mu)) \end{aligned}$$

and

$$\begin{aligned} & e^{-\nu} + e^{-\bar{\nu}} \frac{g' e^{\bar{\lambda}}(\bar{a}c\sigma(\bar{\mu}) + \bar{c}a\sigma(-\mu)) + e^{-\nu}e^{\lambda}(\bar{a}c\sigma(-\bar{\mu}) + \bar{c}a\sigma(\mu))}{1 - e^{-(\nu+\bar{\nu})}} \\ & - e^{-\nu-\bar{\nu}} \frac{\bar{a}' e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + \bar{j}g\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + \bar{j}g\sigma(\bar{\mu}))}{1 - e^{-(\nu+\bar{\nu})}} \\ & = e^{-\bar{\nu}} e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})). \end{aligned}$$

Finally, by adding above two equations, we get the result.  $\square$

If we define the following three cross-ratios

$$\mathbb{X}_1 := \mathbb{X}_1(\alpha, \gamma) = [r_\gamma, a_\alpha, r_\alpha, a_\gamma],$$

$$\mathbb{X}_2 := \mathbb{X}_2(\alpha, \gamma) = [r_\gamma, r_\alpha, a_\alpha, a_\gamma],$$

$$\mathbb{X}_3 := \mathbb{X}_3(\alpha, \gamma) = [r_\gamma, r_\alpha, a_\gamma, a_\alpha],$$

where  $a_\alpha$  and  $r_\alpha$  are attracting and repelling fixed points of  $\alpha$  respectively and they are the same for  $\gamma$ , by easy calculation (See Lemma 6.2 in [15]), we get  $\mathbb{X}_1 = j\bar{a}$ ,  $\mathbb{X}_2 = c\bar{g}$ ,  $\mathbb{X}_3 = \frac{cg}{aj}$ .

Now we are ready to calculate  $G(P)$  and  $G^r(P)$ .

$$\begin{aligned}
G(P) &= \log[\alpha^+, \gamma^-, \alpha^-, \beta^+] \\
&= \log[\infty, Q(0), 0, R(\infty)] \\
&= \log \frac{\langle \infty, Q(0) \rangle \langle 0, R(\infty) \rangle}{\langle \infty, R(\infty) \rangle \langle 0, Q(0) \rangle} \\
&= \log \frac{\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ f \\ j \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} a' \\ d' \\ g' \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a' \\ d' \\ g' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} c \\ f \\ j \end{pmatrix} \right\rangle} \\
&= \log \frac{\bar{j}\bar{a}'}{\bar{g}'\bar{c}} \\
&= \log \frac{\bar{j}}{\bar{c}} \frac{e^{-\bar{\lambda}}(e^{\bar{\mu}-\mu} + \bar{a}j\sigma(-\bar{\mu}) + \bar{c}g\sigma(\mu)) + e^{-\bar{\nu}}e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})) - e^{\nu-\bar{\nu}} - e^{-\nu}}{e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu}))} \\
&= \log \frac{e^{-\bar{\lambda}}(e^{\bar{\mu}-\mu} + \mathbb{X}_1\sigma(-\bar{\mu}) + \overline{\mathbb{X}_2}\sigma(\mu)) + e^{-\bar{\nu}}e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + \overline{\mathbb{X}_2}\sigma(-\mu) + \mathbb{X}_1\sigma(\bar{\mu})) - e^{\nu-\bar{\nu}} - e^{-\nu}}{e^{-\bar{\lambda}}(\overline{\mathbb{X}_2}\sigma(\mu) + \mathbb{X}_1\overline{\mathbb{X}_3}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(\overline{\mathbb{X}_2}\sigma(-\mu) + \mathbb{X}_1\overline{\mathbb{X}_3}\sigma(\bar{\mu}))}.
\end{aligned}$$

For the last line, we use that  $\frac{\bar{c}j\bar{g}}{j} = \mathbb{X}_1\overline{\mathbb{X}_3}$ . Furthermore,

$$\begin{aligned}
G^r(P) &= \log[\alpha^+, \beta^+, \alpha^-, \beta^-] \\
&= \log[\infty, R(\infty), 0, R(0)] \\
&= \log \frac{< \infty, R(\infty) > < 0, R(0) >}{< \infty, R(0) > < 0, R(\infty) >} \\
&= \log \frac{\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} a' \\ d' \\ g' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} c' \\ f' \\ j' \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c' \\ f' \\ j' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} a' \\ d' \\ g' \end{pmatrix} \right\rangle} \\
&= \log \frac{\frac{g'c'}{j'a'}}{\frac{g'}{a'} \frac{g'(e^{\bar{\lambda}}(\bar{a}c\sigma(\bar{\mu}) + \bar{c}a\sigma(-\mu)) + e^{-\nu}e^{\lambda}(\bar{a}c\sigma(-\bar{\mu}) + \bar{c}a\sigma(\mu)))}{a'(e^{-\lambda}(\bar{g}j\sigma(\bar{\mu}) + \bar{j}g\sigma(-\mu)) + e^{-\nu}e^{-\bar{\lambda}}(\bar{g}j\sigma(-\bar{\mu}) + \bar{j}g\sigma(\mu)))}} \\
&= \log \frac{|g'|^2(e^{\bar{\lambda}}(\bar{a}c\sigma(\bar{\mu}) + \bar{c}a\sigma(-\mu)) + e^{-\nu}e^{\lambda}(\bar{a}c\sigma(-\bar{\mu}) + \bar{c}a\sigma(\mu)))}{|a'|^2(e^{-\lambda}(\bar{g}j\sigma(\bar{\mu}) + \bar{j}g\sigma(-\mu)) + e^{-\nu}e^{-\bar{\lambda}}(\bar{g}j\sigma(-\bar{\mu}) + \bar{j}g\sigma(\mu)))} \\
&= \log \left[ \frac{e^{\bar{\lambda}}(\bar{a}c\sigma(\bar{\mu}) + \bar{c}a\sigma(-\mu)) + e^{-\nu}e^{\lambda}(\bar{a}c\sigma(-\bar{\mu}) + \bar{c}a\sigma(\mu))}{e^{-\lambda}(\bar{g}j\sigma(\bar{\mu}) + \bar{j}g\sigma(-\mu)) + e^{-\nu}e^{-\bar{\lambda}}(\bar{g}j\sigma(-\bar{\mu}) + \bar{j}g\sigma(\mu))} \right. \\
&\quad \cdot \left. \frac{|e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu}))|^2}{|e^{-\bar{\lambda}}(e^{\bar{\mu}-\mu} + \bar{a}j\sigma(-\bar{\mu}) + \bar{c}g\sigma(\mu)) + e^{-\bar{\nu}}e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})) - e^{\nu-\bar{\nu}} - e^{-\nu}|^2} \right] \\
&= \log \left[ \frac{e^{\bar{\lambda}}(\bar{a}c\sigma(\bar{\mu}) + \bar{c}a\sigma(-\mu)) + e^{-\nu}e^{\lambda}(\bar{a}c\sigma(-\bar{\mu}) + \bar{c}a\sigma(\mu))}{e^{-\lambda}(\bar{g}j\sigma(\bar{\mu}) + \bar{j}g\sigma(-\mu)) + e^{-\nu}e^{-\bar{\lambda}}(\bar{g}j\sigma(-\bar{\mu}) + \bar{j}g\sigma(\mu))} \right. \\
&\quad \cdot \left. \frac{e^{-\bar{\lambda}}(g\bar{j}\sigma(\mu) + j\bar{g}\sigma(-\bar{\mu})) + e^{-\bar{\nu}}e^{-\lambda}(g\bar{j}\sigma(-\mu) + j\bar{g}\sigma(\bar{\mu}))}{|e^{-\bar{\lambda}}(e^{\bar{\mu}-\mu} + \bar{a}j\sigma(-\bar{\mu}) + \bar{c}g\sigma(\mu)) + e^{-\bar{\nu}}e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + g\bar{c}\sigma(-\mu) + j\bar{a}\sigma(\bar{\mu})) - e^{\nu-\bar{\nu}} - e^{-\nu}|^2} \right].
\end{aligned}$$

Here, the denominator is

$$|e^{-\bar{\lambda}}(e^{\bar{\mu}-\mu} + \mathbb{X}_1\sigma(-\bar{\mu}) + \overline{\mathbb{X}_2}\sigma(\mu)) + e^{-\bar{\nu}}e^{\bar{\lambda}}(e^{\mu-\bar{\mu}} + \overline{\mathbb{X}_2}\sigma(-\mu) + \mathbb{X}_1\sigma(\bar{\mu})) - e^{\nu-\bar{\nu}} - e^{-\nu}|^2$$

because  $\mathbb{X}_1 = j\bar{a}$  and  $\mathbb{X}_2 = c\bar{g}$ , and the numerator is

$$\begin{aligned}
& ((e^{\bar{\lambda}}\sigma(\bar{\mu}) + e^{-\nu}e^{\lambda}\sigma(-\bar{\mu}))\bar{a}c + (e^{\bar{\lambda}}\sigma(-\mu) + e^{-\nu}e^{\lambda}\sigma(\mu))\bar{c}a) \\
& \cdot ((e^{-\bar{\lambda}}\sigma(\mu) + e^{-\bar{\nu}}e^{-\lambda}\sigma(-\mu))g\bar{j} + (e^{-\bar{\lambda}}\sigma(-\bar{\mu}) + e^{-\bar{\nu}}e^{-\lambda}\sigma(\bar{\mu}))j\bar{g}) \\
& = |\sigma(\mu)(1 - e^{-\nu}e^{-\mu}e^{\lambda-\bar{\lambda}})|^2 \bar{a}c g\bar{j} \\
& + \sigma(\bar{\mu})^2 (1 - e^{-\nu}e^{-\mu}e^{\lambda-\bar{\lambda}})(e^{-\bar{\nu}}e^{\bar{\lambda}-\lambda} - e^{-\mu})\bar{a}c j\bar{g} \\
& + \sigma(\mu)^2 \overline{(1 - e^{-\nu}e^{-\mu}e^{\lambda-\bar{\lambda}})(e^{-\bar{\nu}}e^{\bar{\lambda}-\lambda} - e^{-\mu})} \bar{a}c g\bar{j} \\
& + |\sigma(\mu)(e^{-\bar{\nu}}e^{\bar{\lambda}-\lambda} - e^{-\mu})|^2 \bar{a}c j\bar{g} \\
& = |\sigma(\mu)(1 - e^{-\nu}e^{-\mu}e^{\lambda-\bar{\lambda}})|^2 |\mathbb{X}_1|^2 \mathbb{X}_3 + |\sigma(\mu)(e^{-\bar{\nu}}e^{\bar{\lambda}-\lambda} - e^{-\mu})|^2 |\mathbb{X}_1|^2 \overline{\mathbb{X}_3} \\
& + 2Re[\sigma(\bar{\mu})^2 (1 - e^{-\nu}e^{-\mu}e^{\lambda-\bar{\lambda}})(e^{-\bar{\nu}}e^{\bar{\lambda}-\lambda} - e^{-\mu})\mathbb{X}_1\mathbb{X}_2]
\end{aligned}$$

because  $\bar{a}c g\bar{j} = |\mathbb{X}_1|^2 \mathbb{X}_3$ . We can also express  $G(P)$  and  $G^r(P)$  as a function of  $\mathbb{X}_1, \mathbb{X}_2, \lambda, \mu, \nu$  and  $tr[A, B]$  because  $\mathbb{X}_3$  can be written as a function of  $\mathbb{X}_1, \mathbb{X}_2, \lambda, \mu, \nu$  and  $tr[A, B]$  by Corollary 6.5 in [15]. Furthermore, by Proposition 7.6 in [15], it is also possible to write  $G(P)$  and  $G^r(P)$  as a function of  $\lambda, \mu, \nu, tr[A, B], tr(AB)$  and  $tr(A^{-1}B)$ .

**5.2.2. Cusp Gap functions.** In this section, we calculate cusp gap functions in fuchsian case, which has already done in [8] but here we use complex hyperbolic coordinates and give a new proof.

When  $\alpha$  represents a cusp, we normalize so that the fixed point of  $\alpha$ , say  $\alpha^+$ , is  $\infty$ . In fuchsian case, all fixed points of  $\alpha, \beta$ , and  $\gamma$  are on the  $t$ -axis in Heisenberg group. Then as a matrix point of view,

$$\alpha = \begin{bmatrix} 1 & 0 & it \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where  $t$  is a real number and  $i = \sqrt{-1}$ . As in the above section, for  $Q, R \in SU(2, 1)$  and  $\mu, \nu \in S$ , we can write

$$\gamma = QE(\mu)Q^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix} \begin{bmatrix} e^{\mu} & 0 & 0 \\ 0 & e^{\bar{\mu}-\mu} & 0 \\ 0 & 0 & e^{-\bar{\mu}} \end{bmatrix} \begin{bmatrix} \bar{j} & \bar{f} & \bar{c} \\ \bar{h} & \bar{e} & \bar{b} \\ \bar{g} & \bar{d} & \bar{a} \end{bmatrix},$$

$$\beta = \gamma^{-1}\alpha^{-1} = RE(\nu)R^{-1} = \begin{bmatrix} a' & b' & c' \\ d' & e' & f' \\ g' & h' & j' \end{bmatrix} \begin{bmatrix} e^{\nu} & 0 & 0 \\ 0 & e^{\bar{\nu}-\nu} & 0 \\ 0 & 0 & e^{-\bar{\nu}} \end{bmatrix} \begin{bmatrix} \bar{j}' & \bar{f}' & \bar{c}' \\ \bar{h}' & \bar{e}' & \bar{b}' \\ \bar{g}' & \bar{d}' & \bar{a}' \end{bmatrix},$$

where  $e^\mu$  and  $e^\nu$  are attracting eigenvalues of  $\gamma$  and  $\beta$  respectively. Then, by a direct calculation, the periods of  $\beta$  and  $\gamma$  are

$$\ell(\beta) = \log[\beta^-, \beta(y), \beta^+, y] = \log e^{\lambda+\bar{\lambda}} = \lambda+\bar{\lambda} \quad \text{and} \quad \ell(\gamma) = \mu+\bar{\mu} \pmod{2\pi i}.$$

Since the fixed points of  $\beta$  and  $\gamma$  are on the  $t$ -axis in Heisenberg group,

$$\gamma^+ = \begin{pmatrix} a \\ d \\ g \end{pmatrix} \sim \begin{pmatrix} it_1 \\ 0 \\ 1 \end{pmatrix}, \quad \gamma^- = \begin{pmatrix} c \\ f \\ j \end{pmatrix} \sim \begin{pmatrix} it_2 \\ 0 \\ 1 \end{pmatrix},$$

where  $t_1$  and  $t_2$  are distinct real numbers. By using some identities from  $QQ^{-1} = I$ , we can show that  $Q$  must be of the form

$$Q = \begin{bmatrix} it_1 g & 0 & it_2 j \\ 0 & -i\bar{g}\bar{j}(t_1 - t_2) & 0 \\ g & 0 & j \end{bmatrix},$$

where  $g\bar{j}(t_1 - t_2) = -i$ ,  $j\bar{g}(t_1 - t_2) = i$ , and  $|gj(t_1 - t_2)| = 1$ . Similarly,

$$R = \begin{bmatrix} is_1 g' & 0 & is_2 j' \\ 0 & -i\bar{g}'\bar{j}'(s_1 - s_2) & 0 \\ g' & 0 & j' \end{bmatrix},$$

where  $s_1$  and  $s_2$  are distinct real numbers (they cannot be the same as  $t_1$  and  $t_2$  as well.) and  $g'\bar{j}'(s_1 - s_2) = -i$ ,  $j'\bar{g}'(s_1 - s_2) = i$ , and  $|g'j'(s_1 - s_2)| = 1$ .

Then,

$$\begin{aligned} \beta &= \gamma^{-1}\alpha^{-1} = QE(-\mu)Q^{-1}\alpha^{-1} \\ &= \begin{bmatrix} it_1 g & 0 & it_2 j \\ 0 & -i\bar{g}\bar{j}(t_1 - t_2) & 0 \\ g & 0 & j \end{bmatrix} \begin{bmatrix} e^{-\mu} & 0 & 0 \\ 0 & e^{\mu-\bar{\mu}} & 0 \\ 0 & 0 & e^{\bar{\mu}} \end{bmatrix} \begin{bmatrix} \bar{j} & 0 & -it_2 \bar{j} \\ 0 & igj(t_1 - t_2) & 0 \\ \bar{g} & 0 & -it_1 \bar{g} \end{bmatrix} \begin{bmatrix} 1 & 0 & -it \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} it_1 g e^{-\mu} & 0 & it_2 j e^{\bar{\mu}} \\ 0 & -i\bar{g}\bar{j}(t_1 - t_2) e^{\mu-\bar{\mu}} & 0 \\ e^{-\mu} g & 0 & e^{\bar{\mu}} j \end{bmatrix} \begin{bmatrix} \bar{j} & 0 & -i\bar{j}(t+t_2) \\ 0 & igj(t_1 - t_2) & 0 \\ \bar{g} & 0 & -i\bar{g}(t+t_1) \end{bmatrix} \\ &= \begin{bmatrix} it_1 e^{-\mu} g \bar{j} + it_2 e^{\bar{\mu}} j \bar{g} & 0 & t_1(t+t_2) e^{-\mu} g \bar{j} + t_2(t+t_1) e^{\bar{\mu}} j \bar{g} \\ 0 & e^{\mu-\bar{\mu}} & 0 \\ e^{-\mu} g \bar{j} + e^{\bar{\mu}} j \bar{g} & 0 & -i(t+t_2) e^{-\mu} g \bar{j} - i(t+t_1) e^{\bar{\mu}} j \bar{g} \end{bmatrix} \\ &= \begin{bmatrix} \frac{e^{-\mu} t_1 - e^{\bar{\mu}} t_2}{t_1 - t_2} & 0 & \frac{t(e^{\bar{\mu}} t_2 - e^{-\mu} t_1) + t_1 t_2 (e^{\bar{\mu}} - e^{-\mu})}{t_1 - t_2} i \\ 0 & e^{\mu-\bar{\mu}} & 0 \\ \frac{e^{\bar{\mu}} - e^{-\mu}}{t_1 - t_2} i & 0 & \frac{t(e^{\bar{\mu}} - e^{-\mu}) + e^{\bar{\mu}} t_1 - e^{-\mu} t_2}{t_1 - t_2} \end{bmatrix}. \end{aligned}$$

Furthermore, a direct calculation of  $\beta$  is

$$\begin{aligned}
\beta &= RE(\nu)R^{-1} \\
&= \begin{bmatrix} is_1g' & 0 & is_2j' \\ 0 & -i\overline{g'j'}(s_1 - s_2) & 0 \\ g' & 0 & j' \end{bmatrix} \begin{bmatrix} e^\nu & 0 & 0 \\ 0 & e^{\overline{\nu}-\nu} & 0 \\ 0 & 0 & e^{-\overline{\nu}} \end{bmatrix} \begin{bmatrix} \overline{j'} & 0 & -is_2\overline{j'} \\ 0 & ig'j'(s_1 - s_2) & 0 \\ \overline{g'} & 0 & -is_1\overline{g'} \end{bmatrix} \\
&= \begin{bmatrix} is_1e^\nu g'\overline{j'} + is_2e^{-\overline{\nu}}j'\overline{g'} & 0 & s_1s_2(e^\nu g'\overline{j'} + e^{-\overline{\nu}}j'\overline{g'}) \\ 0 & e^{\overline{\nu}-\nu} & 0 \\ e^\nu g'\overline{j'} + e^{-\overline{\nu}}j'\overline{g'} & 0 & -is_2e^\nu g'\overline{j'} - is_1e^{-\overline{\nu}}j'\overline{g'} \end{bmatrix} \\
&= \begin{bmatrix} \frac{s_1e^\nu - s_2e^{-\overline{\nu}}}{s_1 - s_2} & 0 & \frac{e^{-\overline{\nu}} - e^\nu}{s_1 - s_2} s_1s_2i \\ 0 & e^{\overline{\nu}-\nu} & 0 \\ \frac{e^{-\overline{\nu}} - e^\nu}{s_1 - s_2} i & 0 & \frac{s_1e^{-\overline{\nu}} - s_2e^\nu}{s_1 - s_2} \end{bmatrix}.
\end{aligned}$$

By comparing entries of  $\beta$ , we get the following five identities.

$$\begin{aligned}
(1) \quad & \frac{e^{-\mu}t_1 - e^{\overline{\mu}}t_2}{t_1 - t_2} = \frac{s_1e^\nu - s_2e^{-\overline{\nu}}}{s_1 - s_2} \\
(2) \quad & \frac{e^{\overline{\mu}} - e^{-\mu}}{t_1 - t_2} = \frac{e^{-\overline{\nu}} - e^\nu}{s_1 - s_2} \\
(3) \quad & e^{\mu-\overline{\mu}} = e^{\overline{\nu}-\nu} \\
(4) \quad & \frac{t(e^{\overline{\mu}}t_2 - e^{-\mu}t_1) + t_1t_2(e^{\overline{\mu}} - e^{-\mu})}{t_1 - t_2} = \frac{e^{-\overline{\nu}} - e^\nu}{s_1 - s_2} s_1s_2 \\
(5) \quad & \frac{t(e^{\overline{\mu}} - e^{-\mu}) + e^{\overline{\mu}}t_1 - e^{-\mu}t_2}{t_1 - t_2} = \frac{s_1e^{-\overline{\nu}} - s_2e^\nu}{s_1 - s_2}
\end{aligned}$$

By adding the first equation and the last one, we get

$$\frac{t(e^{\overline{\mu}} - e^{-\mu})}{t_1 - t_2} + e^{\overline{\mu}} + e^{-\mu} = e^\nu + e^{-\overline{\nu}}.$$

Hence,

$$t_1 - t_2 = \frac{(e^{\overline{\mu}} - e^{-\mu})t}{e^\nu + e^{-\overline{\nu}} - e^{\overline{\mu}} - e^{-\mu}}.$$

If we substitute it to the second equation,

$$s_1 - s_2 = \frac{e^{-\overline{\nu}} - e^\nu}{e^{\overline{\mu}} - e^{-\mu}}(t_1 - t_2) = \frac{e^{-\overline{\nu}} - e^\nu}{e^{\overline{\mu}} - e^{-\mu}} \cdot \frac{(e^{\overline{\mu}} - e^{-\mu})t}{e^\nu + e^{-\overline{\nu}} - e^{\overline{\mu}} - e^{-\mu}} = \frac{(e^{-\overline{\nu}} - e^\nu)t}{e^\nu + e^{-\overline{\nu}} - e^{\overline{\mu}} - e^{-\mu}}.$$

Now we subtract (1) from (5), then

$$\frac{(t + t_1 + t_2)(e^{\overline{\mu}} - e^{-\mu})}{t_1 - t_2} = \frac{(s_1 + s_2)(e^{-\overline{\nu}} - e^\nu)}{s_1 - s_2}.$$

Here, we substitute  $t_1 - t_2$  and  $s_1 - s_2$ , then we get  $t + t_1 + t_2 = s_1 + s_2$ .  
Hence

$$\begin{aligned} t &= s_1 + s_2 - t_1 - t_2 \\ &= s_1 + s_1 - \frac{(e^{-\bar{\nu}} - e^{\nu})t}{e^{\nu} + e^{-\bar{\nu}} - e^{\bar{\mu}} - e^{-\mu}} - \frac{(e^{\bar{\mu}} - e^{-\mu})t}{e^{\nu} + e^{-\bar{\nu}} - e^{\bar{\mu}} - e^{-\mu}} - t_2 - t_2, \end{aligned}$$

so

$$(e^{\nu} + e^{-\bar{\nu}} - e^{\bar{\mu}} - e^{-\mu} + e^{-\bar{\nu}} - e^{\nu} + e^{\bar{\mu}} - e^{-\mu})t = 2(e^{\nu} + e^{-\bar{\nu}} - e^{\bar{\mu}} - e^{-\mu})(s_1 - t_2) \quad (*)$$

Now let's calculate the cusp gap function

$$W(P) = W_{\alpha}(\gamma^-, \beta^+) = \frac{\partial_y \log[\alpha^+, \gamma^-, y, \beta^+]}{\partial_y \log[\alpha^+, s_0, y, \alpha(s_0)]} \Big|_{y=\alpha^+}.$$

Here  $\alpha^+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ ,  $\gamma^- = \begin{pmatrix} it_2 \\ 0 \\ 1 \end{pmatrix}$ , and  $\beta^+ = \begin{pmatrix} is_1 \\ 0 \\ 1 \end{pmatrix}$ . Since  $W(P)$  is independent of  $s_0$ , we substitute  $s_0 = \gamma^-$  and since  $y$  is a point on boundary, we let  $y = \begin{pmatrix} ix \\ 0 \\ 1 \end{pmatrix}$ , where  $x$  is a real number, then  $y = \alpha^+$  means  $x = \infty$ .



$$\begin{aligned}
W(P) &= W_\alpha(\gamma^-, \beta^+) = \frac{\partial_y \log[\alpha^+, \gamma^-, y, \beta^+]}{\partial_y \log[\alpha^+, s_0, y, \alpha(s_0)]} \Big|_{y=\alpha^+} \\
&= \frac{\partial_y \log \frac{\langle \alpha^+, \gamma^- \rangle \langle y, \beta^+ \rangle}{\langle \alpha^+, \beta^+ \rangle \langle y, \gamma^- \rangle}}{\partial_y \log \frac{\langle \alpha^+, s_0 \rangle \langle y, \alpha(s_0) \rangle}{\langle \alpha^+, \alpha(s_0) \rangle \langle y, s_0 \rangle}} \Big|_{y=\alpha^+} \\
&= \frac{\partial_y [\log \langle \alpha^+, \gamma^- \rangle + \log \langle y, \beta^+ \rangle - \log \langle \alpha^+, \beta^+ \rangle - \log \langle y, \gamma^- \rangle]}{\partial_y [\log \langle \alpha^+, \gamma^- \rangle + \log \langle y, \alpha(\gamma^-) \rangle - \log \langle \alpha^+, \alpha(\gamma^-) \rangle - \log \langle y, \gamma^- \rangle]} \Big|_{y=\alpha^+} \\
&= \frac{\partial_x [\log \left\langle \begin{pmatrix} ix \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} is_1 \\ 0 \\ 1 \end{pmatrix} \right\rangle - \log \left\langle \begin{pmatrix} ix \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} it_2 \\ 0 \\ 1 \end{pmatrix} \right\rangle]}{\partial_x [\log \left\langle \begin{pmatrix} ix \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} i(t+t_2) \\ 0 \\ 1 \end{pmatrix} \right\rangle - \log \left\langle \begin{pmatrix} ix \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} it_2 \\ 0 \\ 1 \end{pmatrix} \right\rangle]} \Big|_{x=\infty} \\
&= \frac{\partial_x [\log(ix - is_1) - \log(ix - it_2)]}{\partial_x [\log(ix - i(t+t_2)) - \log(ix - it_2)]} \Big|_{x=\infty} \\
&= \frac{\frac{1}{x-s_1} - \frac{1}{x-t_2}}{\frac{1}{x-(t+t_2)} - \frac{1}{x-t_2}} \Big|_{x=\infty} \\
&= \frac{\frac{s_1-t_2}{(x-s_1)(x-t_2)}}{\frac{t}{(x-t-t_2)(x-t_2)}} \Big|_{x=\infty} \\
&= \frac{s_1 - t_2}{t}.
\end{aligned}$$

By the equation (\*),

$$\begin{aligned}
W(P) &= \frac{s_1 - t_2}{t} = \frac{e^{-\bar{\nu}} - e^{-\mu}}{e^\nu + e^{-\bar{\nu}} - e^{\bar{\mu}} - e^{-\mu}} = \frac{1}{\frac{e^\nu + e^{-\bar{\nu}} - e^{\bar{\mu}} - e^{-\mu}}{e^{-\bar{\nu}} - e^{-\mu}}} = \frac{1}{1 + \frac{e^\nu - e^{\bar{\mu}}}{e^{-\bar{\nu}} - e^{-\mu}}} \\
&= \frac{1}{1 + \frac{e^\nu - e^{\mu+\nu-\bar{\nu}}}{e^{-\bar{\nu}} - e^{-\mu}}} \quad (\because e^{\bar{\mu}} = e^{\mu+\nu-\bar{\nu}}) \\
&= \frac{1}{1 + \frac{e^{\mu+\nu}(e^{-\mu} - e^{-\bar{\nu}})}{e^{-\bar{\nu}} - e^{-\mu}}} \\
&= \frac{1}{1 - e^{\mu+\nu}} = \frac{1}{1 - e^{\frac{\mu+\bar{\mu}+\nu+\bar{\nu}}{2}}} \\
&= \frac{1}{1 - e^{\frac{\mu+\bar{\mu}}{2}} e^{\frac{\nu+\bar{\nu}}{2}}} \\
&= \frac{1}{1 - e^{\frac{\ell(\beta)+2n\pi i}{2}} e^{\frac{\ell(\gamma)+2m\pi i}{2}}},
\end{aligned}$$

where  $m$  and  $n$  are integers. Here, we note that for  $A_1, A_2, A_3 \in SL(2, \mathbb{C})$ ,  $tr(A_i) = \pm 2 \cosh(\frac{\ell(A_i)}{2})$  and  $tr(A_1)tr(A_2)tr(A_3) < 0$ .

Hence, among  $tr(A_i)$  for  $i = 1, 2, 3$ , either only one has negative value or all have negative values.

By similar argument, since  $\ell(\alpha) = 0$  in our case, either  $\cosh(\frac{\ell(\beta)}{2})$  or  $\cosh(\frac{\ell(\gamma)}{2})$  must be negative. Hence, among  $m$  and  $n$ , one is even and the other is odd, so

$$W(P) = \frac{1}{1 - e^{\frac{\ell(\beta)+2n\pi i}{2}} e^{\frac{\ell(\gamma)+2m\pi i}{2}}} = \frac{1}{1 + e^{\frac{\ell(\beta)+\ell(\gamma)}{2}}}.$$

Similarly, we can show that

$$\begin{aligned} W^r(P) &= W_\alpha(\beta^+, \beta^-) = \frac{s_2 - s_1}{t} \\ &= \frac{e^\nu - e^{-\bar{\nu}}}{e^\nu + e^{-\bar{\nu}} - e^{\bar{\mu}} - e^{-\mu}} \\ &= \frac{\sinh \frac{\ell(\beta)}{2}}{\cosh \frac{\ell(\gamma)}{2} + \cosh \frac{\ell(\beta)}{2}}. \end{aligned}$$

## REFERENCES

- [1] H. Akiyoshi, H. Miyachi and M. Sakuma, Variations of McShane's identity for punctured surface groups, in Spaces of Kleinian groups, London Math. Soc. Lecture Note series, vol 329, Cambridge Univ. Press. Cambridge, 2006, 151-185.
- [2] J. Birman and C. Series, Geodesics with bounded intersection are sparse, Topology, 24 (1985), 217-225.
- [3] M. Bourdon, Structure conforme au bord et flot géodésique d'un CAT(-1)-espace, Enseign. Math. (2) 41 (1995), no. 1-2, 63-102.
- [4] B. H. Bowditch, *A proof of McShane's identity via Markoff triples*, Bull. London Math. Soc. **28** (1996), no. 1, 73-78.
- [5] B. H. Bowditch, *A variation of McShane's identity for once-punctured torus bundles*, Topology **36** (1997), no. 2, 325-334.
- [6] W. M. Goldman, Complex hyperbolic Geometry, Oxford Univ. Press, (1999).
- [7] I. Kim, Marked length Rigidity of rank one symmetric spaces and their product, Topology 40 (2001), no.6, 1295-1323.
- [8] F. Labourie and G. McShane, Cross ratios and identities for Higher Teichmüller-Thurston Theory, Duke Math 148 (9) (2009), 279-345.
- [9] G. McShane, *A remarkable identity for lengths of curves*, Ph.D. Thesis, University of Warwick, 1991.
- [10] G. McShane, *Simple geodesics and a series constant over Teichmuller space*, Invent. Math. **132** (1998), no. 3, 607-632.
- [11] M. Mirzakhani, Simple geodesics and Weil-Peterson volumes of moduli spaces of bordered Riemann surfaces, Inv. Math. 167 (1) (2007), 179-222.

- [12] M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves*. J. Amer. Math. Soc. 20 (2007), no. 1, 123
- [13] M. Mirzakhani, *Growth of the number of simple closed geodesics on hyperbolic surfaces*. Ann. of Math. (2) 168 (2008), no. 1, 97125.
- [14] G. D. Mostow, Strong rigidity of locally symmetric spaces, Ann. of Math. Stud., vol. 78. Princeton Univ. Press, Princeton, NJ, 1973.
- [15] J. R. Parker, I. D. Platis, Complex Hyperbolic Fenchel-Nielsen coordinates, Topology 47 (2008), no.2, 101-135.
- [16] S.P. Tan, Y.L. Wong and Y. Zhang, *Generalizations of McShane's identity to hyperbolic cone-surfaces*, J. Differential Geom. **72** (2006), no. 1, 73–112.
- [17] S.P. Tan, Y.L. Wong and Y. Zhang, *McShane's identity for classical Schottky groups*, Pacific Journal of Math **237** (2008), no. 1, 183–200.
- [18] S.P. Tan, Y.L. Wong and Y. Zhang, Generalized Markoff maps and McShane's identity, Adv. Math 217 (2008), 761-813.
- [19] S.P. Tan, Y.L. Wong and Y. Zhang, Delambre-Gauss formulas for augmented, right-angled hexagons in hyperbolic 4-space, preprint (2011).

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